Coefficient estimates of functions in the class concerning with spirallike functions (Extensions of the historical calculus transforms in the geometric function theory)

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Abstract

For analytic functions $f(z)$ normalized by $f(0)=0$ and $f'(0)=1$ in the open unit disk $\mathbb{U}$, a new subclass $\mathcal{S}_{\alpha}$ of $f(z)$ concerning with spirallike functions in $\mathbb{U}$ is introduced. The object of the present paper is to discuss an extremal function for the class $\mathcal{S}_{\alpha}$ and coefficient estimates of functions $f(z)$ belonging to the class $\mathcal{S}_{\alpha}$.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n} \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C};\ |z|<1\}$.
If $f(z)\in\mathcal{A}$ satisfies the following inequality

$$\mathrm{Re}\left(\frac{1}{\alpha}\frac{zf'(z)}{f(z)}\right)>1 \qquad (z\in\mathbb{U}) \tag{1.2}$$

for some complex number α ($|\alpha-\frac{1}{2}|<\frac{1}{2}$), then we say that $f(z)\in\mathcal{S}_{\alpha}$. If $\alpha=|\alpha|e^{i\varphi}$, then the condition (1.2) is equivalent to

$$\mathrm{Re}\left(e^{-i\varphi}\frac{zf'(z)}{f(z)}\right)>|\alpha| \qquad (z\in\mathbb{U}).$$

Therefore, we note that a function $f(z)\in\mathcal{S}_{\alpha}$ is spirallike in $\mathbb{U}$ which implies that $f(z)$ is univalent in $\mathbb{U}$.

Further, if $0<\alpha<1$, then $f(z)\in\mathcal{S}_{\alpha}$ is starlike of order α (cf. Robertson[3]).
Let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

$$p(z)=1+\sum_{k=1}^{\infty}c_{k}z^{k} \tag{1.3}$$

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which are analytic in $\mathbb{U}$ and satisfy

$$\operatorname{Re} p(z) > 0 \qquad (z \in \mathbb{U}).$$

Then we say that $p(z) \in \mathcal{P}$ is the Carathéodory function (cf. Caratéodory [1] or Duren [2]).

Remark 1.1 Let us consider a function $f(z) \in \mathcal{A}$ which satisfies

$$\left|\frac{f(z)}{zf'(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \qquad (z \in \mathbb{U}) \tag{1.4}$$

for $|\alpha - \frac{1}{2}| < \frac{1}{2}$. If we write that $F(z) = \frac{zf'(z)}{f(z)}$, then the inequality (1.4) can be written by

$$\left|\frac{2\alpha - F(z)}{F(z)}\right| < 1 \qquad (z \in \mathbb{U}).$$

This implies that

$$\alpha\overline{F(z)} + \bar{\alpha}F(z) > 2|\alpha|^2 \qquad (z \in \mathbb{U}).$$

It follows that

$$\left(\frac{F(z)}{\alpha}\right) + \overline{\left(\frac{F(z)}{\alpha}\right)} > 2 \qquad (z \in \mathbb{U}).$$

Therefore, the inequality (1.4) is equivalent to

$$\operatorname{Re}\left(\frac{1}{\alpha}\frac{zf'(z)}{f(z)}\right) > 1 \qquad (z \in \mathbb{U}).$$

2 Coefficient estimates

In this section, we discuss the coefficient estimates of a_n for $f(z) \in \mathcal{S}_\alpha$. To establish our results, we need the following lemma due to Carathéodory [1].

Lemma 2.1 *If a function $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$ satisfies the following inequality*

$$\operatorname{Re} p(z) > 0 \qquad (z \in \mathbb{U}),$$

then

$$|c_k| \leqq 2 \qquad (k = 1, 2, 3, \cdots)$$

with equality for

$$p(z) = \frac{1+z}{1-z}.$$

Now, we introduce the following theorem.

Theorem 2.2 *Extremal function for the class $\mathcal{S}_\alpha$ is $f(z)$ defined by*

$$f(z) = \frac{z}{(1-z)^{2\alpha(\operatorname{Re}(\frac{1}{\alpha})-1)}}. \tag{2.1}$$

Proof. From the definition of the class $\mathcal{S}_\alpha$, we have that

$$\mathrm{Re}\left(\frac{1}{\alpha}\frac{zf'(z)}{f(z)} - 1\right) > 0.$$

Moreover, it is clear that

$$\mathrm{Re}\left(\frac{1}{\alpha}\right) > 1 \qquad \left(|\alpha - \frac{1}{2}| < \frac{1}{2}\right).$$

Then, if the function $F(z)$ is defined by

$$F(z) = \frac{\frac{1}{\alpha}\frac{zf'(z)}{f(z)} - 1 - i\mathrm{Im}\left(\frac{1}{\alpha}\right)}{\mathrm{Re}\left(\frac{1}{\alpha}\right) - 1},$$

we see that

$$\mathrm{Re}F(z) > 0 \;\; and \;\; F(0) = 1,$$

so that, $F(z) \in \mathcal{P}$.
Therefore, Lemma 2.1 shows us that

$$F(z) = \frac{\frac{1}{\alpha}\frac{zf'(z)}{f(z)} - 1 - i\mathrm{Im}\left(\frac{1}{\alpha}\right)}{\mathrm{Re}\left(\frac{1}{\alpha}\right) - 1} = \frac{1+z}{1-z}.$$

It follows that,

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = 2\alpha\left(\mathrm{Re}\left(\frac{1}{\alpha}\right) - 1\right)\frac{1}{1-z}.$$

Integrating both sides from 0 to z on t, we have that

$$\int_0^z \left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right)dt = 2\alpha\left(\mathrm{Re}\left(\frac{1}{\alpha}\right) - 1\right)\int_0^z \frac{1}{1-t}dt,$$

which implies that

$$\log\frac{f(z)}{z} = \log\frac{1}{(1-z)^{2\alpha(\mathrm{Re}(\frac{1}{\alpha})-1)}}.$$

Therefore, we obtain that

$$f(z) = \frac{z}{(1-z)^{2\alpha(\mathrm{Re}(\frac{1}{\alpha})-1)}}.$$

This is the extremal function of the class $\mathcal{S}_\alpha$. □

Next, we discuss the coefficient estimates of $f(z)$ belonging to the class $\mathcal{S}_\alpha$.

Theorem 2.3 *If a function $f(z) \in \mathcal{S}_\alpha$, then*

$$|a_n| \leqq \frac{1}{(n-1)!}\prod_{k=1}^{n-1}(2(\cos(\arg(\alpha)) - |\alpha|) + (k-1)) \qquad (n = 2, 3, 4 \cdots).$$

Equality holds true for $f(z)$ given by (2.1).

Proof. By using same method with Theorem 2.2 , if we set $F(z)$ that

$$F(z) = \frac{\frac{1}{\alpha}\frac{zf'(z)}{f(z)} - 1 - i\mathrm{Im}\left(\frac{1}{\alpha}\right)}{\mathrm{Re}\left(\frac{1}{\alpha}\right) - 1}, \tag{2.2}$$

then it is clear that $F(z) \in \mathcal{P}$.
Letting

$$F(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

Lemma 2.1 gives us that

$$|c_m| \leqq 2 \quad (m = 1, 2, 3 \cdots).$$

Now, from (2.2),

$$\left(\mathrm{Re}(\frac{1}{\alpha}) - 1\right)F(z) = \frac{1}{\alpha}\frac{zf'(z)}{f(z)} - 1 - i\mathrm{Im}(\frac{1}{\alpha}).$$

Let $\mathrm{Re}(\frac{1}{\alpha}) - 1 = s$ and $1 + i\mathrm{Im}(\frac{1}{\alpha}) = A$.
This implies that

$$(\alpha s F(z) + \alpha A) f(z) = z f'(z).$$

Then, the coefficients of z^n in both sides lead to

$$n a_n = (\alpha s + \alpha A) a_n + \alpha s (a_{n-1} c_1 + a_{n-2} c_2 + \cdots + a_{n-r} c_r + \cdots + a_2 c_{n-2} + c_{n-1}).$$

Therefore, we see that

$$a_n = \frac{\alpha s}{n - \alpha s - \alpha A}(a_{n-1} c_1 + a_{n-2} c_2 + \cdots + a_{n-r} c_r + \cdots + a_2 c_{n-2} + c_{n-1}).$$

This shows that

$$\begin{aligned}
|a_n| &= \frac{|\alpha(\mathrm{Re}(\frac{1}{\alpha}) - 1)|}{|n - \alpha(\mathrm{Re}(\frac{1}{\alpha}) - 1) - \alpha(1 + i\mathrm{Im}(\frac{1}{\alpha}))|}|a_{n-1} c_1 + a_{n-2} c_2 + \cdots + a_{n-r} c_r + \cdots + a_2 c_{n-2} + c_{n-1}| \\
&= \frac{\cos(\arg(\alpha)) - |\alpha|}{n-1}|a_{n-1} c_1 + a_{n-2} c_2 + \cdots + a_{n-r} c_r + \cdots + a_2 c_{n-2} + c_{n-1}| \\
&\leqq \frac{\cos(\arg(\alpha)) - |\alpha|}{n-1}(|a_{n-1}||c_1| + |a_{n-2}||c_2| + \cdots + |a_{n-r}||c_r| + \cdots + |a_2||c_{n-2}| + |c_{n-1}|) \\
&\leqq \frac{\cos(\arg(\alpha)) - |\alpha|}{n-1}(2|a_{n-1}| + 2|a_{n-2}| + \cdots + 2|a_2| + 2) \\
&\leqq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{n-1}\sum_{k=1}^{n-1}|a_k| \qquad (|a_1| = 1).
\end{aligned}$$

To prove that

$$|a_n| \leqq \frac{1}{(n-1)!}\prod_{k=1}^{n-1}(2(\cos(\arg(\alpha)) - |\alpha|) + (k-1)),$$

we need to show that

$$|a_n| \leqq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{n-1}\sum_{k=1}^{n-1}|a_k| \leqq \frac{1}{(n-1)!}\prod_{k=1}^{n-1}(2(\cos(\arg(\alpha)) - |\alpha|) + (k-1)). \tag{2.3}$$

We use the mathematical induction for the proof.
When $n = 2$, this assertion is true.
We assume that the proposition is true for $n = 2, 3, 4, \cdots, m-1$.
For $n = m$, we obtain that

$$\begin{aligned}
|a_m| &\leqq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1}\sum_{k=1}^{m-1}|a_k| \\
&= \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1}\left(\sum_{k=1}^{m-2}|a_k| + |a_{m-1}|\right) \\
&= \frac{m-2}{m-1}\frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-2}\sum_{k=1}^{m-2}|a_k| + \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1}|a_{m-1}| \\
&\leqq \frac{m-2}{(m-1)!}\prod_{k=1}^{m-2}(2(\cos(\arg(\alpha)) - |\alpha|) + k - 1) \\
&\quad + \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1}\frac{1}{(m-2)!}\prod_{k=1}^{m-2}(2(\cos(\arg(\alpha)) - |\alpha|) + k - 1) \\
&= \frac{1}{(m-1)!}\prod_{k=1}^{m-2}(2(\cos(\arg(\alpha)) - |\alpha|) + k - 1)\left(m - 2 + 2(\cos(\arg(\alpha)) - |\alpha|)\right) \\
&= \frac{1}{(m-1)!}\prod_{k=1}^{m-1}(2(\cos(\arg(\alpha)) - |\alpha|) + k - 1).
\end{aligned}$$

Thus the inequality (2.3) is true for $n = m$. By the mathematical induction, we prove that

$$|a_n| \leqq \frac{1}{(n-1)!}\prod_{k=1}^{n-1}(2(\cos(\arg(\alpha)) - |\alpha|) + (k-1)) \qquad (n = 2, 3, 4 \cdots).$$

For the equality, we consider the extremal function $f(z)$ given by Theorem 2.2. Since

$$f(z) = \frac{z}{(1-z)^{2\alpha(\mathrm{Re}(\frac{1}{\alpha})-1)}},$$

if we let

$$2\alpha(\mathrm{Re}(\frac{1}{\alpha}) - 1) = j,$$

then $f(z)$ becomes that

$$f(z) = z(1-z)^{-j} = z\left(\sum_{n=0}^{\infty}\binom{-j}{n}(-z)^n\right) = z + \sum_{n=2}^{\infty}\frac{j(j+1)\cdots(j+n-2)}{(n-1)!}z^n.$$

From the above, we obtained

$$a_n = \frac{1}{(n-1)!}\prod_{k=1}^{n-1}\left(2\alpha(\mathrm{Re}(\frac{1}{\alpha}) - 1) + k - 1\right).$$

For $n = 2$,

$$|a_2| = 2|\alpha||\mathrm{Re}(\frac{1}{\alpha}) - 1| = 2(\cos(\arg(\alpha)) - |\alpha|).$$

Furthermore, for $n \geqq 3$, we have that

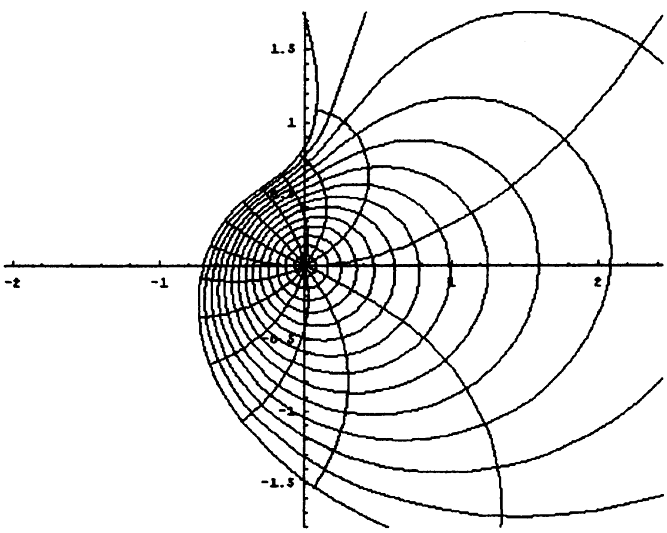
$$\begin{aligned} |a_n| &= \left|\frac{1}{(n-1)!}\prod_{k=1}^{n-1}(2\alpha(\mathrm{Re}(\frac{1}{\alpha}) - 1) + k - 1)\right| \\ &= \frac{1}{(n-1)!}\prod_{k=1}^{n-1}|2\alpha(\mathrm{Re}(\frac{1}{\alpha}) - 1) + k - 1| \\ &\leqq \frac{1}{(n-1)!}\prod_{k=1}^{n-1}(2(\cos(\arg(\alpha)) - |\alpha|) + k - 1). \end{aligned}$$

Equality holds true for some real α $(0 < \alpha < 1)$.
This completes the proof of Theorem 2.3 . □

Example 2.4 Let $\alpha = \frac{1}{2} + \frac{1}{4}i$ in (2.1). Then we have that

$$f(z) = \frac{z}{(1-z)^{\frac{6+3i}{10}}}.$$

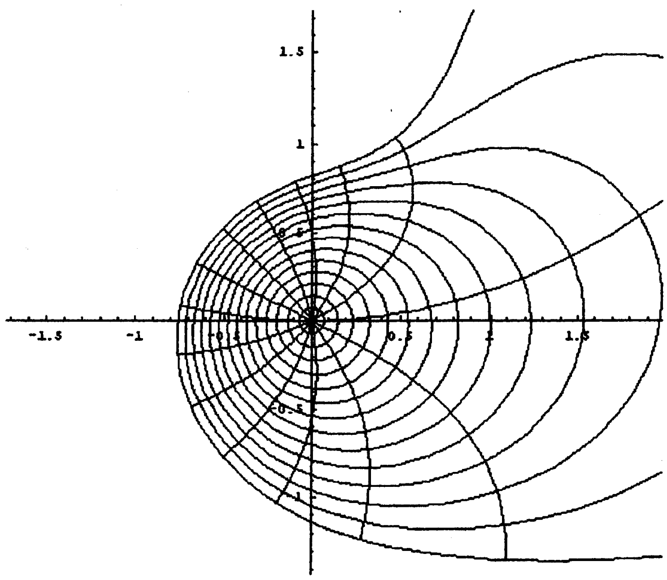
This function $f(z)$ maps the unit disk $\mathbb{U}$ onto the following domain.



Example 2.5 If we take $\alpha = \frac{2}{3} + \frac{1}{4}i$ in (2.1), then we have that

$$f(z) = \frac{z}{(1-z)^{\frac{184+69i}{438}}}.$$

This function $f(z)$ maps the unit disk $\mathbb{U}$ onto the following domain.



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